A Network Model for Transmission Lines with Gyromagnetic Coupling

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Abstract—Reciprocal ensembles of coupled transmission lines have been studied for many years using matrix techniques. In this paper, the lossless multiconductor transmission line model is extended to permit a description of gyromagnetic coupling effects. The novel ingredient which allows such an extension is the incorporation of distributed gyrators into the elemental line-length prototype. These gyrators provide antireciprocal coupling between the ensemble conductors. The amount of coupling is expressed by a geometry-dependent factor which in effect measures a given structure relative to an ideal Faraday rotator in the same medium. The gyromagnetic coupling factor, in conjunction with the derived expressions for mode propagation factors and characteristic impedances, provides a means of interpolating between the known limits of no coupling and ideal Faraday rotation. General relations are derived for two-line systems and for symmetrical, quasi-TEM three-line systems.

Introduction

THE TRANSMISSION-LINE model is one of the oldest and most useful tools of the designer of microwave networks. While it may seem a crude approximation to the intricacies of the electromagnetic field distribution in any given problem situation, its very simplicity allows a keener insight into the gross behavior of such networks. Furthermore, microwave network problems are usually concerned only with the behavior of sets of scattering parameters, and not with the field distributions per se. Therefore, it is frequently most expedient to formulate the scattering properties of a junction as a more or less complicated transmission-line problem, solving for the various propagation factors and characteristic impedances by one of the well-known integral equation methods leading to iterative, perturbational, or variational formulas.

The appearance of microwave ferrites on the scene in recent years has, however, led to a rather awkward situation. "Exact" methods (i.e., boundary-value problem formulations) of solving propagation problems in magnetized ferrites have been of value in a number of important but limited cases. These have usually amounted to determining the propagation factor and possibly also the characteristic impedance of single, isolated lines. As ferrite component technology has matured, circuits have been built using two or more coupled lines in a medium partially or completely filled with ferrite to obtain nonreciprocal effects. Analysis of this type of configuration has so far been restricted to special cases, such as those treated by Boyet and Seidel[1] and by Jones, Matthaei, and Cohn,[2] using perturbational methods.

The present paper is the result of an attempt to generalize the matrix transmission-line problem in a way that allows for gyromagnetic coupling between conductors in an ensemble. The model is essentially an heuristic one, chosen solely because it appears to be the simplest such model which fits the limiting cases properly. The novel ingredient which allows nonreciprocal coupling to occur is the introduction of distributed gyrators into the elemental prototype of transmission-line length. Chief emphasis is placed on an exploration of the characteristics of the two-line system with gyromagnetic coupling only and with mixed reciprocal and nonreciprocal coupling. Because the symmetrical three-line ensemble is of interest in circulator applications, a short discussion of its characteristics is included.

Essentials of the Matrix Transmission Line Problem

Since the nonreciprocal model to be discussed presently is a generalization of the conventional matrix transmission-line problem, it is appropriate to review here briefly the nature of this problem. For the simplest case of a single, isolated, uniform transmission line, it is customary to define an elemental prototype section of length dz containing a distributed shunt admittance and a distributed excitation. Analysis of this prototype section leads to the familiar differential equations,

$$V' = -j\omega LI \tag{1}$$

$$I' = -j\omega CV. \tag{2}$$

Equations (1) and (2) are then combined to give second-order differential equations in V and I of the form

$$V'' + k^2 V = 0 \tag{3}$$

$$I'' + k^2 I = 0 \tag{4}$$

where $k^2 = \omega^2 LC$. Assuming solutions of the form $V = \hat{V}e^{j\beta z}$ and $I = \hat{I}e^{j\beta z}$ leads to the characteristic equation

$$k^2 - \beta^2 = 0 \tag{5}$$

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[1] H. Boyet and H. Seidel, "Analysis of nonreciprocal effects in an n-wire ferrite-loaded transmission line," *Proc. IRE*, vol. 45, pp. 491–495, April 1957.

[2] E. M. T. Jones, G. L. Matthaei, and S. B. Cohn, "A nonreciprocal, TEM-mode structure for wide-band gyrator and isolator applications," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-7, pp. 453–460; October 1959.

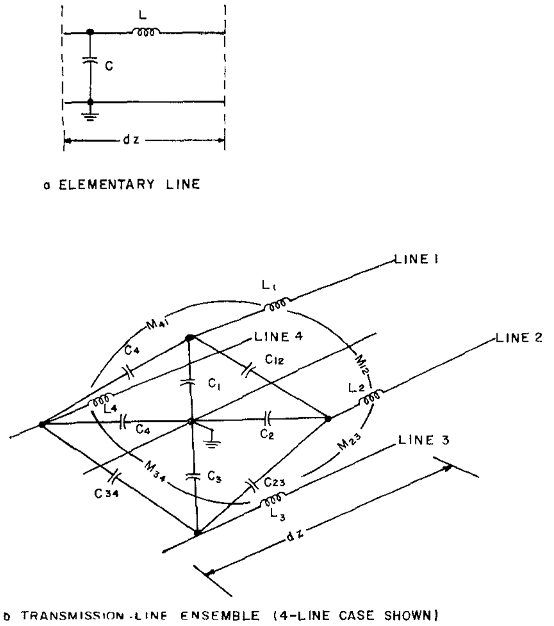


Fig. 1. Reciprocal transmission-line prototype sections.

with solutions $\beta = \pm k$. Using this result in conjunction with either (1) or (2) allows the ratio of V to I to be prescribed for any root of (5) in the form of a "characteristic impedance,"

$$\frac{\hat{V}}{\hat{I}} = Z_0 = \frac{\omega L}{\beta} = \frac{\beta}{\omega C}. \tag{6}$$

Applying the results of (5) and (6) to the assumed form of solutions for V and I, plus load and generator end conditions then prescribes the total z dependence of the phasors $\hat{V}$ and $\hat{I}$.

All of the previous information is, of course, completely familiar to engineering students at the undergraduate level. What is perhaps not so familiar is the generalization of this problem to ensembles of lossless, uniform transmission lines. In this latter case, the ordinary differential equation is replaced by a system of differential equations, and the characteristic (5) goes over into a matrix eigenvalue equation. As before, the starting point is an elemental section of line length dz, having series distributed self and mutual inductances, and shunt distributed capacities to ground and between lines. Equations (1) and (2) then become systems of equations, which can be written in matrix notation

$$\boldsymbol{V}' = -j\omega \boldsymbol{L}\boldsymbol{I} \tag{7}$$

$$\boldsymbol{I}' = -j\omega \boldsymbol{C}\boldsymbol{V}. \tag{8}$$

Here $\boldsymbol{V}$ and $\boldsymbol{I}$ are column vectors of voltage and current, with each member of the columns representing the voltage (with respect to a reference ground) and current on a particular line. Similarly $\boldsymbol{V}'$ and $\boldsymbol{I}'$ are columns of dV/dz and dI/dz for the same lines, respectively. The quantities $\boldsymbol{L}$ and $\boldsymbol{C}$ are square matrices of generalized series inductance and shunt capacity, derived from the elemental prototype section of line length. Since the prototype is that of lossless network, it follows that $\boldsymbol{L}$ and $\boldsymbol{C}$ are nonsingular Hermitian matrices, becoming real and symmetric for the reciprocal case. Equations (7) and (8) are now combined in a manner formally identical with the combining of (1) and (2) to give the matrix differential equations

$$\boldsymbol{V}'' + \boldsymbol{K}^2\boldsymbol{V} = 0 \tag{9}$$

$$\boldsymbol{I}'' + \boldsymbol{K}^{2\dagger}\boldsymbol{I} = 0 \tag{10}$$

with $\boldsymbol{K}^2 = \omega^2\boldsymbol{L}\boldsymbol{C}$, and the dagger indicating Hermitian conjugation of the matrix. Assuming solutions of the form $\boldsymbol{V} = \hat{\boldsymbol{V}}e^{j\beta z}$ and $\hat{\boldsymbol{I}} = \boldsymbol{I}e^{j\beta z}$ then leads to the matrix eigenvalue equations

$$(\boldsymbol{K}^2 - \beta^2\boldsymbol{E})\hat{\boldsymbol{V}} = (\boldsymbol{K}^{2\dagger} - \beta^2\boldsymbol{E})\hat{\boldsymbol{I}} = 0 \tag{11}$$

where $\boldsymbol{E}$ is the unit matrix. The quantities β^2 are seen to be eigenvalues of the matrices $\boldsymbol{K}^2$ or $\boldsymbol{K}^{2\dagger}$, the results being identical for either choice. In general, the number of distinct values of β^2 may be as large as the rank of $\boldsymbol{K}^2$, but it frequently happens that degeneracies occur. Finally, since $\boldsymbol{L}$ and $\boldsymbol{C}$ are both Hermitian, the values of β^2 must be real.

Nondegenerate values of β^2 are, of course, associated with particular modes of voltage-current distribution which propagate as traveling waves with a uniquely defined propagation factor. These modal distributions are given by the eigenvectors $\boldsymbol{v}_i$ of voltage and $\boldsymbol{i}_i$ of current, associated with the ith value of β^2, and defined by

$$\boldsymbol{K}^2\boldsymbol{v}_i = \beta_i^2\boldsymbol{v}_i \tag{12}$$

and

$$\boldsymbol{K}^{2\dagger}\boldsymbol{i}_i = \beta_i^2\boldsymbol{i}_i. \tag{13}$$

The eigenvectors $\boldsymbol{v}_i$ and $\boldsymbol{i}_i$ are chosen normalized so that both vectors have equivalent length, satisfying a mode power orthogonality relationship of the form

$$\boldsymbol{i}_i^\dagger\boldsymbol{v}_j = \delta_{ij} \tag{14}$$

where δ is the Kronecker delta. The mode voltage and current vectors are, of course, multiplied by complex amplitude coefficients U_i and J_i to be determined from end conditions. The amplitude of the current vector is related to that of the voltage vector through a mode characteristic impedance Z_i in a manner analogous to the single-line case,

$$U_i = Z_iJ_i. \tag{15}$$

This impedance can be determined (along with $\boldsymbol{v}_i$ if desired) by substituting $\boldsymbol{i}_i$ and β_i into a form of (17),

$$U_i\boldsymbol{v}_i = \frac{\omega}{\beta_i}\boldsymbol{L}\boldsymbol{i}_i. \tag{16}$$

Because J_i has been taken as unity on the right-hand side of the previous expression, it follows from (15) that

$U_i = Z_i$ on the left-hand side. Then multiplying by $\boldsymbol{i}_i^\dagger$ and applying (14)

$$Z_i = \frac{\omega}{\beta_i} \boldsymbol{i}_i^\dagger \boldsymbol{L} \boldsymbol{i}_i. \tag{17}$$

Of course, it is possible to work through (8) and obtain the reciprocal of Z_i, the mode characteristic admittance Y_i,

$$Y_i = \frac{\omega}{\beta_i} \boldsymbol{v}_i^\dagger \boldsymbol{C} \boldsymbol{v}_i. \tag{18}$$

Note that since $\boldsymbol{i}_i^\dagger \boldsymbol{L} \boldsymbol{i}_i$ and $\boldsymbol{v}_i^\dagger \boldsymbol{C} \boldsymbol{v}_i$ are Hermitian forms and hence equal to real numbers, the characteristic immitances Z_i and Y_i are either pure real or pure imaginary, depending on the nature of β_i.

When degenerate values of β^2 occur, a certain flexibility will exist in the choice of the eigenvectors. However, care must be exercised to ensure that mode orthogonality in the power sense is obtained. Thus, an acceptable method of generating n eigenvectors of current and voltage for an n-fold degeneracy of β^2 is to select one voltage or current vector arbitrarily and calculate its partner and characteristic impedance using (17) or (18) as appropriate. A second eigenvector of the original type is then constructed to be orthogonal to the first partner, and the process repeated until n eigenvectors of both types and n characteristic impedances are obtained.

Essential Form of the Model

Transmission lines coupled in a magnetized ferrite medium have been observed to behave in a nonreciprocal manner. Therefore, it is clear that the strictly reciprocal case must be modified to agree with the more general behavior. Not just any modification will do, of course; the resulting model must have characteristics which are permissible on the basis of network theory and which are in accord with the field theoretical results for propagation in magnetized ferrites in certain limiting cases. There are two such limiting cases of importance here, 1), the TEM-limit case in which no Faraday rotation is permitted and 2) the TEM case in which no constraint is imposed on Faraday rotation. For the first situation, Suhl and Walker[3] have shown that the medium takes on an effective permeability μ_e given by

$$\mu_e = 2 \frac{\mu_+ \mu_-}{\mu_+ + \mu_-} \cdot \tag{19}$$

Here μ_+ and μ_- are eigenvalues of the transverse Polder tensor for a longitudinally magnetized infinite ferrite medium:

$$\mu_+ = \mu + \kappa \tag{20}$$

$$\mu_- = \mu - \kappa. \tag{21}$$

The second case is the classic situation in which a linearly polarized plane wave propagates in a longitudinally magnetized infinite ferrite medium. Under these circumstances it is appropriate to choose two orthogonal axes in the transverse plane and define the system as a pair of transmission lines whose excitations are proportional to the projections of the electric field vector on the respective axes. These two equivalent lines are coupled only through the "mutual permeability" terms in the transverse Polder tensor itself,

$$[\mu] = \begin{bmatrix} \mu & -j\kappa \\ j\kappa & \mu \end{bmatrix}. \tag{22}$$

In both of these cases, and throughout all that follows, it is assumed that the ferrite and conductors behave as a lossless system, implying that the ferrite is not operated in the vicinity of resonance.

To represent a lossless transmission-line system which can be nonreciprocal, it is necessary to modify the properties of the reciprocal series inductance matrix $\boldsymbol{L}$, the reciprocal shunt capacity matrix $\boldsymbol{C}$, or both. Carlin[4,5] has shown that the basic nonreciprocal circuit element is the ideal gyrator, which has an impedance matrix of the form

$$Z_{\text{Gyrator}} = j\omega N \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} \tag{23}$$

where N is a real magnitude factor. Since the gyrator is lossless and perfectly antireciprocal, it should be possible in principal to account for the behavior of a lossless gyrotropic medium in a transmission-line ensemble by incorporating "distributed gyrator" combinations into the elemental line-length prototype. For a gyromagnetic medium such as ferrite, the antireciprocal effect is a function of magnetic field intensity (i.e., line current) and results in induced electric field intensity (line voltage). Consequently, it is appropriate to introduce the distributed gyrators into the series impedance branch only.

Even before introducing distributed gyrators, however, it is necessary to adjust the model of the simple, uncoupled line somewhat in order to bring it into correspondence with the first limiting case of (19). This can be done readily by expressing the effective inductance L_e of the simple series branch as the parallel combination of two branches of inductance $2L_+$ and $2L_-$, respectively. In terms of L_+ and L_-, L_e has the value

$$L_e = 2 \frac{L_+ L_-}{L_+ + L_-} \tag{24}$$

[3] H. Suhl and L. R. Walker, "Topics in guided wave propagation through gyromagnetic media," Bell Telephone System Monograph 2322, pp. 166–173, 1954.

[4] H. J. Carlin, "Principles of gyrator networks," *Proc. Symp. on Mod. Advances in Microwave Tech.*, p. 175, November 1954.

[5] H. J. Carlin, "On the physical realizability of linear nonreciprocal networks," *Proc. IRE*, vol. 43, pp. 608–616; May 1955.

where, for a structure completely filled with ferrite, the relation

$$\frac{L_+}{L_-} = \frac{\mu_+}{\mu_-} \tag{25}$$

is assumed. L_e is thus seen to be a distributed inductance computed in the usual way from the field distribution, using μ_e as the medium permeability for those portions of the structure containing ferrite.

Consider next a symmetrical two-line system which is to be coupled only gyromagnetically. The series impedance portion of the elemental prototype consists, in the absence of coupling, of two sets of parallel branches of $2L_+$ and $2L_-$ distributed inductance. Two additional parallel branches are now added to each line and coupled through distributed gyrators $2N_+$ and $2N_-$. After some algebra, this system can be reduced to an equivalent series-form $\boldsymbol{L}$ matrix suitable for use in (17),

$$\boldsymbol{L} = \frac{1}{\frac{1}{L_e^2} - \frac{1}{N_e^2}} \begin{bmatrix} \frac{1}{L_e} & j\frac{1}{N_e} \\ -j\frac{1}{N_e} & \frac{1}{L_e} \end{bmatrix} \tag{26}$$

where

$$N_e = 2\frac{N_+N_-}{N_+ + N_-}\,. \tag{27}$$

Suppose now, that

$$N_+ = \frac{L_+}{\sigma} \tag{28}$$

$$N_- = -\frac{L_-}{\sigma} \tag{29}$$

where σ is a coupling efficiency factor which varies from zero (no coupling) to unity (maximum coupling). Under this assumption,

$$N_e = \frac{2}{\sigma}\,\frac{L_+L_-}{(L_+ - L_-)}\,. \tag{30}$$

Now, substituting for L_e and N_e in terms of σ, L_+ and L_- in (26),

$$\boldsymbol{L} = \frac{2L_+L_-}{(L_+ + L_-)^2 - \sigma^2(L_+ - L_-)^2} \times \begin{bmatrix} L_+ + L_- & -j\sigma(L_+ - L_-) \\ j\sigma(L_+ - L_-) & L_+ + L_- \end{bmatrix}. \tag{31}$$

The following auxiliary parameters are now defined, for reasons which will become obvious:

$$L_\mu = \tfrac{1}{2}(L_+ + L_-) \tag{32}$$

$$L_\kappa = \tfrac{1}{2}(L_+ - L_-). \tag{33}$$

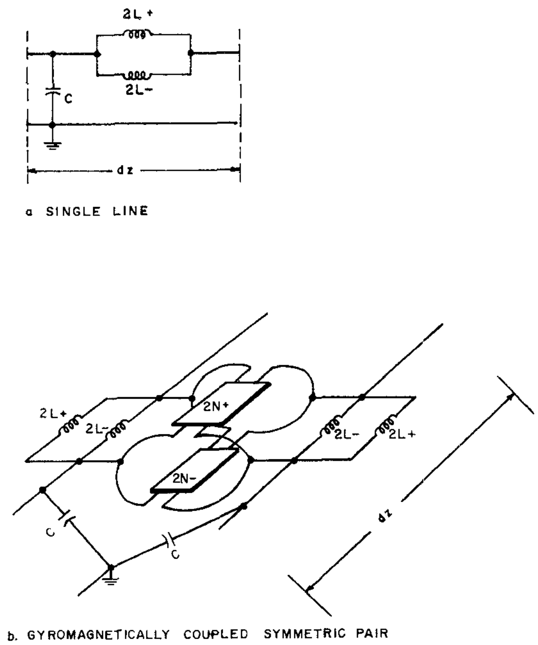


Fig. 2. Prototype sections for transmission lines in a gyromagnetic medium.

Evidently, in a completely filled structure,

$$\frac{L_\kappa}{L_\mu} = \frac{L_+ - L_-}{L_+ + L_-} = \frac{\mu_+ - \mu_-}{\mu_+ + \mu_-} = \frac{\kappa}{\mu}\,. \tag{34}$$

Rewriting (31) in terms of L_μ and L_κ then gives the result

$$\boldsymbol{L} = \left(\frac{L_\mu^2 - L_\kappa^2}{L_\mu^2 - \sigma^2 L_\kappa^2}\right) \begin{bmatrix} L_\mu & -j\sigma L_\kappa \\ j\sigma L_\kappa & L_\mu \end{bmatrix}. \tag{35}$$

Equation (35) is recognized immediately as having the desired limiting behavior when the coupling efficiency parameter σ takes on its extreme values. Thus, (31) shows clearly that $\boldsymbol{L} \rightarrow L_e\boldsymbol{E}$ for $\sigma \rightarrow 0$, while for $\sigma = 1$, (35) becomes

$$\boldsymbol{L} = \begin{bmatrix} L_\mu & -jL_\kappa \\ jL_\kappa & L_\mu \end{bmatrix} \tag{36}$$

which is merely an integrated form of the transverse Polder tensor of (22).

Taking the shunt capacity matrix for this simple case as simply a diagonal form,

$$\boldsymbol{C} = C\boldsymbol{E} \tag{37}$$

it is possible to construct the $\boldsymbol{K}^2$ matrix for general values of σ:

$$\boldsymbol{K}^2 = \omega^2 C\left(\frac{L_\mu^2 - L_\kappa^2}{L_\mu^2 - \sigma^2 L_\kappa^2}\right) \begin{bmatrix} L_\mu & -j\sigma L_\kappa \\ j\sigma L_\kappa & L_\mu \end{bmatrix}. \tag{38}$$

The eigenvalues of this matrix are readily seen to be

$$\beta^2 = \omega^2 C\,\frac{L_\mu^2 - L_\kappa^2}{L_\mu \mp \sigma L_\kappa} = \omega^2 C L_e\left(\frac{1}{1 \mp \zeta}\right) \tag{39}$$

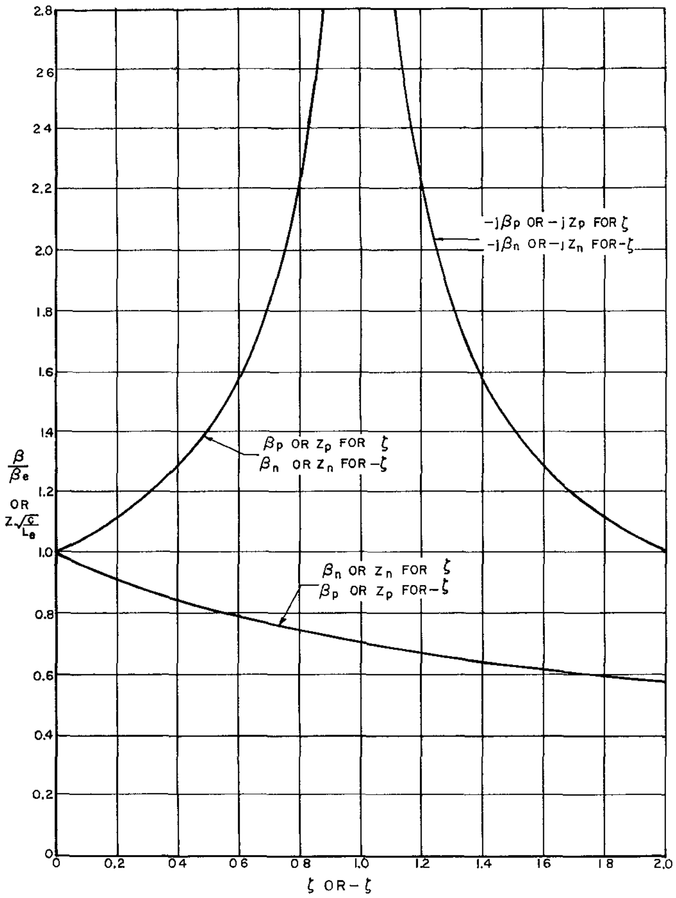


Fig. 3. Mode propagation factor and characteristic impedance variation.

where $\zeta = \sigma(L_\kappa/L_\mu)$. Defining $\beta_e = \omega\sqrt{L_e C}$, the two β magnitudes given by (39) are

$$\beta_p = \beta_e \sqrt{\frac{1}{1-\zeta}} \tag{40}$$

$$\beta_n = \beta_e \sqrt{\frac{1}{1+\zeta}} \cdot \tag{41}$$

Once again, for the completely filled case, where $\zeta = \sigma(\kappa/\mu)$, it is relatively simple to show that $\beta_p \to \beta_+$ and $\beta_n \to \beta_-$ when $\sigma \to 1$.

The eigenvectors associated with the β_p and β_n normal modes can be found using the basic relationships of (12) and (13). Writing $\boldsymbol{K}^2$ in terms of L_e and ζ,

$$\boldsymbol{K}^2 = \frac{\omega^2 L_e C}{1-\zeta^2}\begin{bmatrix} 1 & -j\zeta \\ j\zeta & 1 \end{bmatrix} \tag{42}$$

these become, for the voltage case,

$$\frac{1}{1-\zeta^2}\begin{bmatrix} 1 & -j\zeta \\ j\zeta & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{1 \mp \zeta}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{43}$$

with the identical relation holding for the current eigenvectors. Thus,

$$v_p = i_p = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ j \end{bmatrix} \tag{44}$$

and

$$v_n = i_n = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -j \end{bmatrix} \tag{45}$$

are found to be an appropriate set of eigenvectors for any σ, and the *only* set for $\sigma > 0$.

Clearly, the eigenvectors given by (44) and (45) are in harmony with the requirements of the limiting case $\sigma = 1$. That is, the equal-amplitude, time-quadrature normal modes prescribed previously are precisely in agreement with the known circular polarization normal modes for the infinite-medium Faraday rotator. For intermediate values of σ, the necessity for the particular eigenvectors given is less apparent, but nonetheless intuitively satisfying because of the establishment of opposite senses of circularly polarized in the medium.

The final bit of information required in this simple case is a knowledge of the characteristic impedances of the two normal modes. Applying (17)

$$Z_p = \left\{\sqrt{\frac{L_e}{C}}\,\frac{\sqrt{1-\zeta}}{2(1-\zeta^2)}\right\}\begin{bmatrix} 1 & -j \end{bmatrix}\begin{bmatrix} 1 & -j\zeta \\ j\zeta & 1 \end{bmatrix}\begin{bmatrix} 1 \\ j \end{bmatrix} \tag{46}$$

so that

$$Z_p = \frac{1}{\sqrt{1-\zeta}}\sqrt{\frac{L_e}{C}} \tag{47}$$

similarly,

$$Z_p = \frac{1}{\sqrt{1+\zeta}}\sqrt{\frac{L_e}{C}} \cdot \tag{48}$$

The General Two-Line System

The encouraging results of the previous section suggest that it may be of value to extend the model to more general problems where symmetries do not exist and where reciprocal coupling is permitted in addition to purely antireciprocal coupling. Because of the large number of independent parameters, it is difficult to interpret such general systems for ensembles with a substantial number of lines, to say nothing of the computational labor involved in determining the eigenvalues and eigenvectors. As a result, consideration of the general case will be limited to two-line situations.

Consider then a system of two lines in which one has distributed self-inductances of $2aL_+$ in parallel with $2aL_-$, and the other has $(2/a)L_+$ in parallel with $(2/a)L_-$. To allow for ordinary inductive coupling, distributed mutual inductances of $2pL_+$ and $2pL_-$ will be incorporated into the respective branches of each line. Next, to reflect in some measure the fact that the absolute values of L_+ and L_- are likely to vary with the coupling factor p, normalized values $\hat{L}_+$ and $\hat{L}_-$ will be introduced, defined by

$$\hat{L}_+ = L_+\sqrt{1-p^2} \tag{49}$$

$$\hat{L}_- = L_-\sqrt{1-p^2}\,. \tag{50}$$

Finally, distributed gyrator branches will be added in parallel with the reciprocal branches in the manner of the previous section; since $\hat{L}_+$ and $\hat{L}_-$ are most indicative of the integrated cross-sectional field, the gyrator branches will be given magnitudes of $2(\hat{L}_+/\sigma)$ and $2(\hat{L}_-/\sigma)$.

The "series impedance" portion of the elemental length of each line thus consists of four parallel branches, all of which must, of course, support the same voltage ΔV. Designating the current in each branch by an arbitrary subscript, the appropriate matrix relations for the branches are,

$$-\begin{bmatrix}\Delta V_1\\ \Delta V_2\end{bmatrix} = \frac{j2\omega\hat{L}_+\Delta Z}{\sqrt{1-p^2}}\begin{bmatrix}a & p\\ p & \dfrac{1}{a}\end{bmatrix}\begin{bmatrix}I_{11}\\ I_{12}\end{bmatrix}$$

$$= \frac{j2\omega\hat{L}_-\Delta Z}{\sqrt{1-p^2}}\begin{bmatrix}a & p\\ p & \dfrac{1}{a}\end{bmatrix}\begin{bmatrix}I_{21}\\ I_{22}\end{bmatrix}$$

$$= \frac{2\omega\hat{L}_+\Delta Z}{\sigma}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}I_{31}\\ I_{32}\end{bmatrix}$$

$$= \frac{2\omega\hat{L}_-\Delta Z}{\sigma}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}I_{41}\\ I_{42}\end{bmatrix}. \tag{51}$$

The total currents I_1 and I_2 can be related to the incremental voltages ΔV_1 and ΔV_2 by inverting each of the preceding equations and summing the inverse matrices:

$$\begin{bmatrix}I_1\\ I_2\end{bmatrix} = \frac{-1/\Delta Z}{j2\omega\hat{L}_+\hat{L}_-\sqrt{1-p^2}}\begin{bmatrix}\dfrac{\hat{L}_+ + \hat{L}_-}{a} & -p(\hat{L}_+ + \hat{L}_-) + j\sigma\sqrt{1-p^2}\,(\hat{L}_+ - \hat{L}_-)\\ -p(\hat{L}_+ + \hat{L}_-) - j\sigma\sqrt{1-p^2}\,(\hat{L}_+ - \hat{L}_-) & a(\hat{L}_+ + \hat{L}_-)\end{bmatrix}$$

$$\times\begin{bmatrix}\Delta V_1\\ \Delta V_2\end{bmatrix}. \tag{52}$$

By inverting this expression, the desired $\mathbf{L}$ matrix is found to be

$$\mathbf{L} = \frac{2\hat{L}_+\hat{L}_-/\sqrt{1-p^2}}{(\hat{L}_+ + \hat{L}_-)^2 - \sigma^2(\hat{L}_+ - \hat{L}_-)^2}$$

$$\times\begin{bmatrix}a(\hat{L}_+ + \hat{L}_-) & p(\hat{L}_+ + \hat{L}_-) - j\sigma\sqrt{1-p^2}\,(\hat{L}_+ - L_-)\\ p(\hat{L}_+ + \hat{L}_-) + j\sigma\sqrt{1-p^2}\,(\hat{L}_+ - \hat{L}) & \dfrac{\hat{L}_+ + \hat{L}_-}{a}\end{bmatrix}. \tag{53}$$

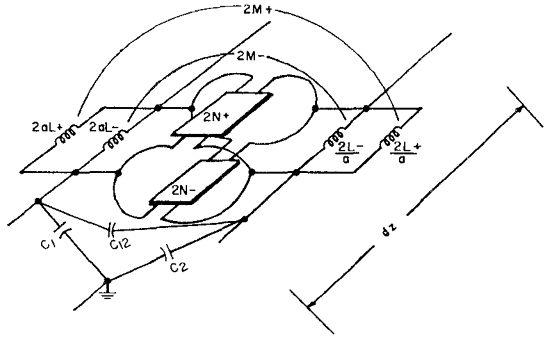


Fig. 4. Prototype section for a general transmission-line pair in a gyromagnetic medium.

As in the previous section, the parameters L_μ and L_κ are now introduced, and after some simplification,

$$\mathbf{L} = \frac{\hat{L}_e}{(1-\zeta^2)\sqrt{1-p^2}}$$

$$\times\begin{bmatrix}a & p - j\zeta\sqrt{1-p^2}\\ p + j\zeta\sqrt{1-p^2} & 1/a\end{bmatrix} \tag{54}$$

where

$$\hat{L}_e = \frac{2\hat{L}_+\hat{L}_-}{\hat{L}_+ + \hat{L}_-} \tag{55}$$

and, as before, the parameter ζ is defined by

$$\zeta = \sigma\frac{\hat{L}_\kappa}{\hat{L}_\mu} = \sigma\frac{\hat{L}_+ - \hat{L}_-}{\hat{L}_+ + \hat{L}_-}\,. \tag{56}$$

A shunt capacitance matrix with distributed coupling is now defined in an analogous manner, i.e.,

$$\mathbf{C} = \frac{\hat{C}}{\sqrt{1-q^2}}\begin{bmatrix}b & -q\\ -q & \dfrac{1}{b}\end{bmatrix}. \tag{57}$$

Now forming $K^2 = \omega^2 LC$,

$$K^2 = \frac{\omega^2 \hat{L}_e \hat{C}}{(1-\zeta^2)\sqrt{(1-p^2)(1-q^2)}} \begin{bmatrix} ab - qp + jq\zeta\sqrt{1-p^2} & -aq + \frac{p}{b} - j\frac{\zeta}{b}\sqrt{1-p^2} \\ \frac{-q}{a} + bp + jb\zeta\sqrt{1-p^2} & \frac{1}{ab} - qp - jq\zeta\sqrt{1-p^2} \end{bmatrix}. \tag{58}$$

The eigenvalues of K^2 are then found to be,

$$\beta^2 = \frac{\omega^2 \hat{L}_e \hat{C}}{(1-\zeta^2)\sqrt{(1-p^2)(1-q^2)}} \times \left\{ \frac{1}{2}\left(ab + \frac{1}{ab}\right) - qp \pm \sqrt{\left[\frac{1}{2}\left(ab + \frac{1}{ab}\right) - qp\right]^2 - (1-p^2)(1-q^2)(1-\zeta^2)} \right\}. \tag{59}$$

Knowing the eigenvalues it is a straightforward but tedious job to compute the eigenvectors and characteristic impedance or admittance for any situation of interest. However, the complexity of the expressions and the large number of independent parameters makes interpretation of results difficult, so that the value of exhibiting these general quantities is questionable. On the other hand, there are a number of particular cases which occur frequently in practice which are simpler, have fewer independent parameters, and are more conducive to gaining insight into the nature of the general problem. Two such cases will be considered now, 1) the geometrically symmetric case and 2) the quasi-TEM case. For each of these cases, the propagation factor expressions will be examined and, for the latter case, the eigenvectors and characteristic impedances will be calculated.

1) The geometrically symmetric case: For this situation the constants a and b become unity. Then from (59)

$$\beta^2 = \frac{\omega^2 \hat{L}_e \hat{C}}{(1-\zeta^2)\sqrt{(1-p^2)(1-q^2)}} \cdot \left\{ (1-qp) \pm \sqrt{(1-qp)^2 - (1-p^2)(1-q^2)(1-\zeta^2)} \right\}. \tag{60}$$

This expression is still rather involved, but can be simplified when q and p do not differ greatly. Thus, take $q = (1+\epsilon)p$, where ϵ is a number sufficiently small so that $\epsilon^2 \ll 2\epsilon$. Then

$$1 - q^2 \approx (1-p^2)\left(1 - \frac{2\epsilon p^2}{1-p^2}\right) \tag{61}$$

$$1 - qp \approx (1-p^2)\left(1 - \frac{\epsilon p^2}{1-p^2}\right). \tag{62}$$

Substituting into (60) after cancelling $a(1-p^2)$ factor,

$$\beta^2 \approx \frac{\omega^2 \hat{L}_e \hat{C}}{(1-\zeta^2)\sqrt{1 - \frac{2\epsilon p^2}{1-p^2}}} \left\{ \left(1 - \frac{\epsilon p^2}{1-p^2}\right) \pm \sqrt{\left(1 - \frac{\epsilon p^2}{1-p^2}\right)^2 - \left(1 - \frac{2\epsilon p^2}{1-p^2}\right)(1-\zeta^2)} \right\} \tag{63}$$

and since

$$1 - \frac{\epsilon p^2}{1-p^2} \approx \sqrt{1 - \frac{2\epsilon p^2}{1-p^2}} \tag{64}$$

a common factor of this sort can be removed, leaving the familiar result

$$\beta^2 \approx \frac{\omega^2 \hat{L}_e \hat{C}}{1 \mp \zeta} = \frac{\beta_e^2}{1 \mp \zeta} \tag{65}$$

which is identical to the elementary case considered in the previous section, implying that the propagation factors are not only unchanged by reciprocal coupling when $q = p$, but are also insensitive to small deviations from this condition. Another case of interest is that in which the reciprocal coupling is essentially of one sort, i.e., $q = 0$ or $p = 0$.

Taking $q = 0$, (60) becomes

$$\beta^2 = \frac{\beta_e^2}{(1-\zeta^2)\sqrt{1-p^2}} \left\{ 1 \pm \sqrt{\zeta^2 + p^2(1-\zeta^2)} \right\}. \tag{66}$$

Replacing p by q, an analogous relation is obtained for $p = 0$. Based on these results, a family of curves of β/β_e has been plotted in Fig. 5.

2) The quasi-TEM case: Here, the characterizing constraints are $q = p$ and $a = 1/b$, so that (59) reduces immediately to the fundamental relation,

$$\beta^2 = \frac{\beta_e^2}{1 \mp \zeta}. \tag{67}$$

Imposing these constraints leads also to a set of simplified eigenvectors appropriate to the quasi-TEM case:

$$v_1 = \frac{1}{\sqrt{2a \cos \theta}} \begin{bmatrix} ae^{j\theta/2} \\ je^{-j\theta/2} \end{bmatrix} \tag{68}$$

$$v_2 = \frac{1}{\sqrt{2a \cos \theta}} \begin{bmatrix} ae^{-j\theta/2} \\ -je^{j\theta/2} \end{bmatrix} \tag{69}$$

$$i_1 = \frac{1}{\sqrt{2a \cos \theta}} \begin{bmatrix} e^{-j\theta/2} \\ jae^{j\theta/2} \end{bmatrix} \tag{70}$$

$$i_2 = \frac{1}{\sqrt{2a \cos \theta}} \begin{bmatrix} e^{j\theta/2} \\ -jae^{-j\theta/2} \end{bmatrix} \tag{71}$$

where $\theta = \sin^{-1} p$.

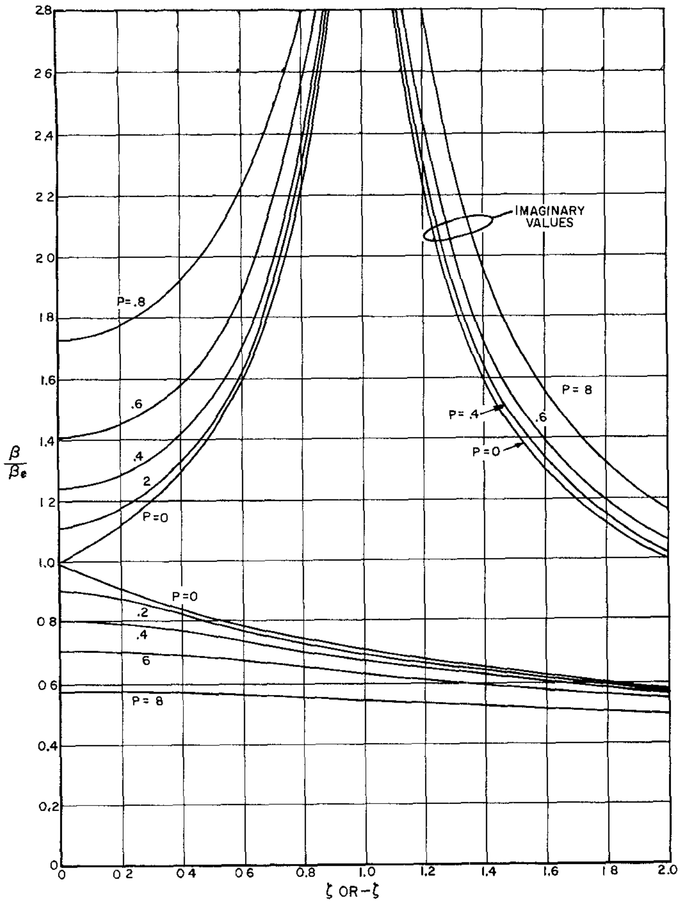


Fig. 5. Influence of reciprocal coupling of only one type on the propagation factor.

The results of (67), (70), and (72) now permit computation of the characteristic impedances Z_1 and Z_2 of the two normal modes using (17). In view of the complexity of the equations, the conclusions are almost startlingly simple, viz:

$$Z_1 = \frac{1}{\sqrt{1-\zeta}}\sqrt{\frac{\hat{L}_e}{\hat{C}}} \tag{72}$$

and

$$Z_2 = \frac{1}{\sqrt{1+\zeta}}\sqrt{\frac{\hat{L}_e}{\hat{C}}}. \tag{73}$$

These expressions are precisely the same as were obtained in the previous section for the elementary case with no reciprocal coupling, the only difference being the replacement of L_e and C by the normalized quantities $\hat{L}_e$ and $\hat{C}$.

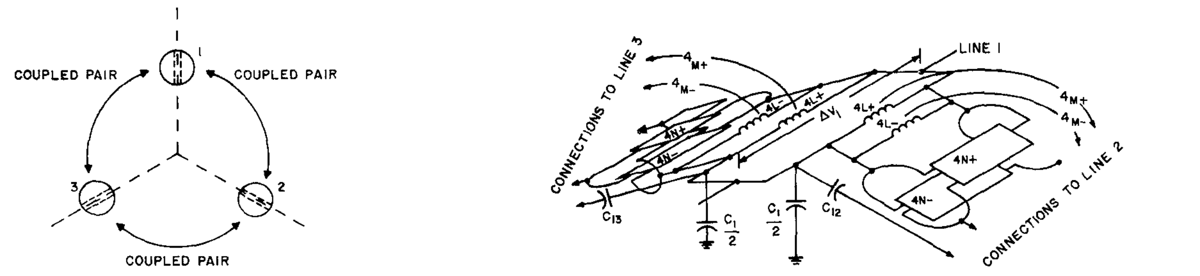
The principal conclusion to be drawn from this examination of the quasi-TEM case is the rather interesting and partially unanticipated way in which reciprocal coupling influences the character of the normal modes. First, the propagation factors for the two modes differ only as a result of the gyromagnetic coupling, and are not directly influenced by reciprocal coupling. Of course, geometries favoring high reciprocal coupling are likely to imply small gyromagnetic coupling. Next, the characteristic impedances of the two modes do not appear to be affected by the extent to which reciprocal coupling is present in a given configuration when normalized values of L_e and C are used. What does change, however, is the nature of the normal mode eigenvectors. In general, the introduction of reciprocal coupling causes the eigenvectors to depart from a "circularly polarized" condition to one which can be considered as representative of an elliptical polarization. As the reciprocal coupling factor increases, the elements of the eigenvectors tend to depart further from a quadrature phase relationship and to approach more nearly an "in-phase" state. The influence of this alteration of eigenvector phase on the behavior of the system under given excitation can, of course, be pronounced because the excitation may resolve itself into rather different amplitudes of the normal modes under different conditions of reciprocal coupling.

Symmetrical Three-Line Systems

Most ferrite applications that involve the use of coupled transmission lines can be conveniently accomplished using only a line pair, and hence, may be sufficiently well described by the previous analysis. One important exception to this broad statement is the coupled-line junction circulator which utilizes a symmetrical, quasi-TEM structure consisting of three transmission lines embedded in a medium which is wholly or partially ferrite. In this section, the general behavior of such a transmission-line system will be investigated as an extension of the two-line system, with a discussion of the detailed circuit properties of the circulator networks reserved for treatment in a later section.

The basic approach to be used in analyzing the three-line case is to consider it as a symmetrical array of quasi-TEM coupled line pairs. That is, in the elemental line length prototype, the series impedance of each line is imagined to consist of two equal impedances in parallel, with coupling to only one of the adjacent lines. This coupling is related to the self-impedance terms in the same way as for a two-line ensemble. The essential nature of this approach is illustrated in Fig. 6. As can readily be seen, the series impedance is composed of a multiplicity of parallel branches, which must be brought to an equivalent series-form L matrix. The first step in the process is to write the relations between the incremental voltage drops ΔV_1, ΔV_2, ΔV_3 and the branch currents. These take the form of three sets of equations each involving two of the incremental voltages. Each of these twelve matrix equations must then be inverted to get relationships which express the branch currents in terms of the voltage increments. The branch currents are then summed to get the total line currents I_1, I_2, and I_3 as functions of the incremental voltage drops.

Finally, this latter relation is inverted to obtain the desired series-form expression for the ΔV's as functions of the line currents and the equivalent inductance matrix is extracted. Carrying out these operations yields the result



a. REPRESENTATION AS THREE COUPLED PAIRS b. PROTOTYPE SECTION COUPLING ELEMENTS FOR LINE 1

Fig. 6. Analysis of the symmetrical three-line system.

$$L = \frac{\hat{L}_e\sqrt{1-p^2}}{(1-p)(2+p)(1-\xi^2)} \times \begin{bmatrix} (2-p)-(2+p)\frac{\xi^2}{3} & p-(2+p)\frac{\xi^2}{3}-2j(1-p)\frac{\xi}{\sqrt{3}} & p-(2+p)\frac{\xi^2}{3}+2j(1-p)\frac{\xi}{\sqrt{3}} \\ p-(2+p)\frac{\xi^2}{3}+2j(1-p)\frac{\xi}{\sqrt{3}} & (2-p)-(2+p)\frac{\xi^2}{3} & p-(2+p)\frac{\xi^2}{3}-2j(1-p)\frac{\xi}{\sqrt{3}} \\ p-(2+p)\frac{\xi^2}{3}-2j(1-p)\frac{\xi}{\sqrt{3}} & p-(2+p)\frac{\xi^2}{3}+2j(1-p)\frac{\xi}{\sqrt{3}} & (2+p)-(2+p)\frac{\xi^2}{3} \end{bmatrix}. \quad (74)$$

The new parameter ξ is defined by the relation

$$\xi = \zeta\,\frac{\sqrt{3(1-p^2)}}{2+p}\,. \quad (75)$$

A capacitance matrix for the three-line ensemble is now required. Fortunately, the capacitance matrices of the constituent two-line arrays are already shunt-form quantities, so that a three-line matrix can be obtained by inspection as a direct sum of the simpler ones:

$$C = \frac{\hat{C}/2}{\sqrt{1-p^2}} \begin{bmatrix} 2 & -p & -p \\ -p & 2 & -p \\ -p & p & 2 \end{bmatrix}. \quad (76)$$

Now forming $K^2 = \omega^2 LC$, a number of common factors cancel, leaving the simple result

$$K^2 = \frac{\omega^2 \hat{L}_e C}{1-\xi^2} \times \begin{Bmatrix} 1-\frac{\xi^2}{3} & -\frac{\xi^2}{3}-j\frac{\xi}{\sqrt{3}} & -\frac{\xi^2}{3}+j\frac{\xi}{\sqrt{3}} \\ -\frac{\xi^2}{3}+j\frac{\xi}{\sqrt{3}} & 1-\frac{\xi^2}{3} & -\frac{\xi^2}{3}-j\frac{\xi}{\sqrt{3}} \\ -\frac{\xi^2}{3}-j\frac{\xi}{\sqrt{3}} & -\frac{\xi^2}{3}+j\frac{\xi}{\sqrt{3}} & 1-\frac{\xi^2}{3} \end{Bmatrix}. \quad (77)$$

Because K^2 is Hermitian, the voltage and current eigenvectors will evidently coincide for each of the normal modes. In addition, the fact that the structure being treated has a rotational symmetry group of order three allows the eigenvectors to be written down immediately as the well known forms,

$$v_1 = i_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (78)$$

$$v_2 = i_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ e^{j2\pi/3} \\ e^{-j2\pi/3} \end{bmatrix} \quad (79)$$

$$v_3 = i_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ e^{-j2\pi/3} \\ e^{j2\pi/3} \end{bmatrix}. \quad (80)$$

Then the eigenvalues β^2 can be found using these eigenvectors in conjunction with (12) and (77) to give the interesting results,

$$\beta_1 = \omega\sqrt{\hat{L}_e\hat{C}} = \beta_e \quad (81)$$

$$\beta_2 = \frac{\beta_e}{\sqrt{1-\xi}} \quad (82)$$

$$\beta_3 = \frac{\beta_e}{\sqrt{1+\xi}}\,. \quad (83)$$

The propagation factors predicted by (81)–(83) are intuitively satisfying in a number of ways. The first eigenvector, for example, corresponds to a normal mode which propagates along the structure with in-phase voltages and currents at every cross-sectional plane. The field conditions existing in the transverse plane are accordingly similar to those of a single line and the propagation factor for this mode should and does correspond to the single-line case. Furthermore, the parameter ξ is seen to be analogous to the parameter ζ in the two-line case, with β_2 and β_3 tending to β_+ and β_- as ξ tends to κ/μ. The particular way in which ξ and ζ are related is also worthy of some comment. In discussing the quasi-TEM two-line system, it was noted that the gyromagnetic coupling parameter ζ did not depend directly on the extent of reciprocal coupling, but rather that the influence of reciprocal coupling was most clearly reflected in a shift of the eigenvector elements away from a time-quadrature condition toward an in-phase state. No such dependence of the eigenvectors of the symmetrical three-line system is possible, however, because the eigenvectors are, in fact, prescribed on the basis of symmetry considerations alone. The influence of reciprocal coupling is felt directly as a reduction of the effective gyromagnetic coupling factor ξ for any particular value of ζ. The dependence of the ξ/ζ ratio on the reciprocal coupling factor p has been plotted in Fig. 7. The normalization factor $\sqrt{3}/2$ which appears in the ξ/ζ ratio term is consistent with the fact that the eigenvector currents occur at a 120° phase relation rather than at the quadrature phase relation leading to maximum coupling. The effective coupling parameter ξ is the important quantity, of course, and depends on the extent of circular polarization generated by the superposition of the magnetic fields produced by the currents in each of the three lines.

Having determined the eigenvectors and eigenvalues of the $\boldsymbol{K}^2$ matrix, it is possible to calculate the characteristic immitances for the three normal modes. Because the $\boldsymbol{L}$ and $\boldsymbol{C}$ matrices possess the same basic symmetry properties as the $\boldsymbol{K}^2$ matrix, the eigenvectors of voltage and current are also eigenvectors of $\boldsymbol{L}$ and $\boldsymbol{C}$, and the Hermitian forms $\boldsymbol{i}^\dagger \boldsymbol{L i}$ and $\boldsymbol{v}^\dagger \boldsymbol{C v}$ of (17) and (18) in fact merely generate the eigenvalues of these matrices. Since the $\boldsymbol{C}$ matrix is the simpler of the two, its eigenvalues will be calculated; they are

$$c_1 = \hat{C}\sqrt{\frac{1-p}{1+p}} \tag{84}$$

$$c_2 = c_3 = \frac{\hat{C}(2+p)}{2\sqrt{1-p^2}} = \hat{C}\frac{\sqrt{3}}{2}\frac{\zeta}{\xi}. \tag{85}$$

The characteristic impedances are then given by

$$Z_1 = \frac{\beta_1}{\omega c_1} = \sqrt{\frac{1+p}{1-p}}\sqrt{\frac{\hat{L}_e}{\hat{C}}} = \frac{1}{v_p c_1} \tag{86}$$

$$Z_2 = \frac{\beta_2}{\omega c_2} = \frac{\frac{2}{\sqrt{3}}\frac{\xi}{\zeta}}{\sqrt{1-\xi}}\sqrt{\frac{\hat{L}_e}{\hat{C}}} = \frac{1}{v_p c_2\sqrt{1-\xi}} \tag{87}$$

$$Z_3 = \frac{\beta_3}{\omega c_3} = \frac{\frac{2}{\sqrt{3}}\frac{\xi}{\zeta}}{\sqrt{1+\xi}}\sqrt{\frac{\hat{L}_e}{\hat{C}}} = \frac{1}{v_p c_2\sqrt{1+\xi}} \tag{88}$$

where v_p is the phase velocity of propagation of the β_1 mode. Clearly, $Z^1\sqrt{\hat{C}/\hat{L}_e}$ is a function only of the parameter p, while the corresponding normalized values $Z_2\sqrt{\hat{C}/\hat{L}_e}$ and $Z_1\sqrt{\hat{C}/\hat{L}_e}$ depend on both p and ξ. The relationship of (86) is accordingly plotted as a single curve in Fig. 8, with those of (87) and (88) taking the form of curves exactly like those of Fig. 3, except that

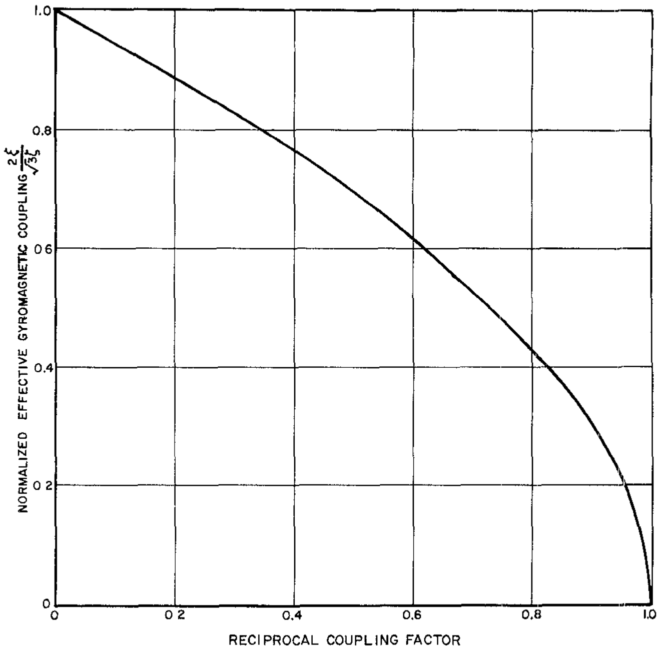


Fig. 7. Reduction of effective gyromagnetic coupling as a result of reciprocal coupling.

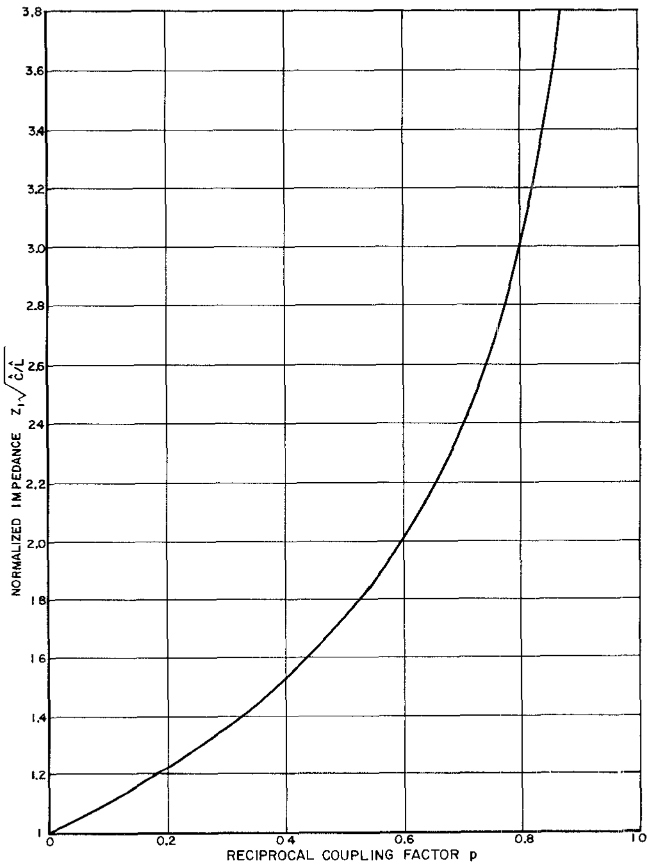


Fig. 8. In-phase mode characteristic impedance variation with reciprocal coupling.

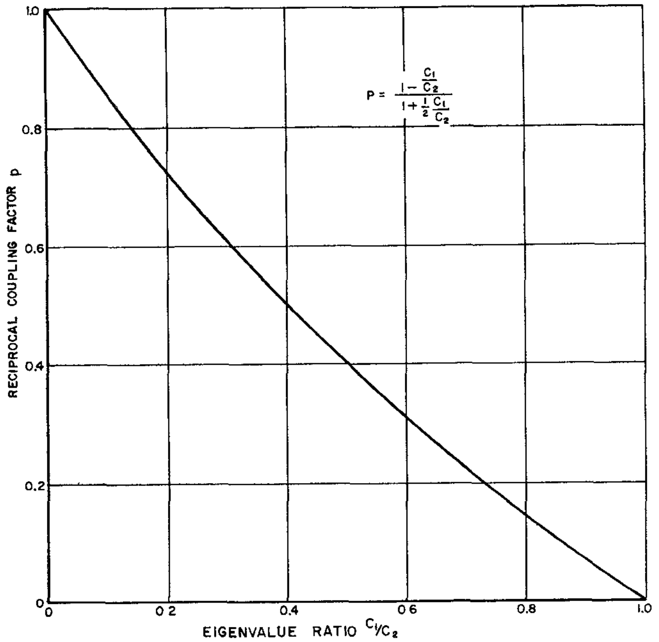


Fig. 9. Relation of the reciprocal coupling factor to the capacitive eigenvalue ratio for a TEM symmetrical three-line.

the abscissa is in units of ξ and the ordinate values are multiplied by the p-dependent factor given in Fig. 7. Finally, the capacitive eigenvalue ratio c_1/c_2 (which is easily measured) has been related to the reciprocal coupling factor p in Fig. 9.

Conclusions

The goal which provided the original impetus to this paper was that of developing a network-theoretical model which was reasonably simple, yet complete enough to describe the sort of behavior observed and reported for transmission-line systems in a ferrite medium. The particular distributed-gyrator combination that has been presented previously was chosen after an examination of several configurations showed it to be best able to describe known limiting cases of ferrite behavior. Like most other network models of physical situations, it is acceptable if it is sufficiently plausible, and useful if it is sufficiently accurate. Assuming plausibility to be reasonably well established, the chief difficulty with this model is likely to be the extent to which accurate computation of the gyromagnetic coupling factors can be carried out. Perhaps the most expedient way of determining the coupling factors for any given geometry will be actually to build a structure and measure its behavior, using the model as a guide to understanding measured results. While this approach may not be very satisfying to the analytically minded, it is accepted practice in large segments of the engineering world.

In common with other network models, certain limitations exist on the applicability of the model which has been developed here. One of the obvious limitations arises out of the restriction of the analysis to lossless lines. Practical ferrites, like practical passive reciprocal networks, always exhibit losses to a greater or lesser degree. Ferrite losses are especially troublesome when the frequency of excitation can couple energy to domains at gyromagnetic resonance. This situation can exist when the ferrite is biased to resonance by the applied magnetic field, when the ferrite is unsaturated at frequencies below $\omega_M = \gamma 4\pi M_s$, or when the applied field is low and a nonlinear high-power loss threshold is exceeded at frequencies below $2\omega_M$. These high-loss regions are undesirable for the sort of conditions where the coupled-line model would apply, and hence must be avoided. Under other conditions, the ferrite losses, while higher than an unfilled line, are generally low enough so that the main effect is that of a slight attenuation of the modes in propagation along the structure. This attenuation, however, is likely to be substantially greater for the mode which couples to the spin system of the ferrite than for the oppositely rotating mode.

Another potential source of difficulty in applying the simple model is the existence, in any given structure, of higher-order modes. If such modes can propagate, the transmission-line ensemble is, in effect, increased in complexity by their number. However, even though a higher-order mode is in a cutoff state, its presence as an evanescent mode of appreciable amplitude may cause substantial modification of the frequency dependence of the ensemble. Such behavior is most likely when the cross-sectional dimensions of the transmission structure begin to be an appreciable fraction of a wavelength (e.g., a half-wavelength) at the operating frequency.

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